Linear Programming (LP)

Simplex Today

- A large variety of Simplex-based algorithms exist to solve LP problems.
- Other (polynomial time) algorithms have been developed for solving LP problems:
  - Khachian algorithm (1979)
  - Kamarkar algorithm (AT&T Bell Labs, mid 80s)
  - See Section 4.10

<u>BUT</u>,

none of these algorithms have been able to beat Simplex in actual practical applications.

<u>HENCE</u>,

Simplex (in its various forms) is and will most likely remain the most dominant LP algorithm for at least the near future

Fundamental Theorem

Extreme point (or Simplex filter) theorem:

If the maximum or minimum value of a linear function defined over a polygonal convex region exists, then it is to be found at the boundary of the

Convex set:

A set (or region) is convex if, for any two points (say, x1 and x2) in that set, the line segment joining these points lies entirely within the set.

A point is by definition convex.

What does the extreme point theorem imply?

- A finite number of extreme points implies a finite number of solutions!

- Hence, search is reduced to a finite set of points

- However, a finite set can still be too large for practical purposes

- Simplex method provides an efficient systematic search guaranteed to converge in a finite number of steps.

Basic Steps of Simplex

1. Begin the search at an extreme point (i.e., a basic feasible solution).

2. Determine if the movement to an adjacent extreme can improve on the optimization of the objective function. If not, the current solution is optimal. If, however, improvement is possible, then proceed to the next step.

3. Move to the adjacent extreme point which offers (or, perhaps, *appears* to offer) the most improvement in the objective function.

4. Continue steps 2 and 3 until the optimal solution is found or it can be shown that the problem is either unbounded or infeasible.

Step 0 – Obtain Canonical Form

IMPORTANT: Simplex only deals with equalities

General Simplex LP model:

min (or max) $z = \Sigma c_i x_i$

s.t.

$A x = b$

$x \geq 0$

In order to get and maintain this form, use

- *slack,* if $x \leq b$, then $x + \text{slack} = b$
- *surplus,* if $x \geq b$, then $x - \text{surplus} = b$
- *artificial variables* (sometimes need to be added to ensure all variables ≥ 0)

Compare constraint conversion with goal conversions using deviation variables

Different "components" of a LP model

- LP model can always be split into a basic and a non-basic part.

- “Transformed” or “reduced” model is another good way to show this.

- This can be represented in mathematical terms as well as in a LP or simplex tableau.

Movement to Adjacent Extreme Point

Given any basis we move to an adjacent extreme point (another basic feasible solution) of the solution space by **exchanging one of the columns that is in the basis for a column that is not in the basis.**

Two things to determine:

1) which (nonbasic) column of A should be brought into the basis so that the solution improves?
2) which column can be removed from the basis such that the solution stays feasible?

Entering and Departing Vector (Variable) Rules

General rules:

- The one non-basic variable to come in is the one which provides the highest reduction in the objective function.
- The one basic variable to leave is the one which is expected to go infeasible first.

NOTE: THESE ARE HEURISTICS!!

Variations on these rules exist, but are rare.

Simplex Variations

Various variations on the simplex method exist:

- "regular" simplex (see Section 4.4)
- two-phase method: Phase I for feasibility and Phase II for optimality (see Section 4.5.1)
- condensed/reduced/revised method: only use the non-basic columns to work with (see Section 4.6)
- (revised) dual simplex (see Section 4.8), etc.

Computational Considerations

- Unrestricted variables (unboundedness)
- Redundancy (linear dependency, modeling errors)
- Degeneracy (some basic variables = 0)
- Round-off errors

Limitations of Simplex

1. Inability to deal with multiple objectives
2. Inability to handle problems with integer variables

Problem 1 is solved using Multiplex

Problem 2 has resulted in:

- Cutting plane algorithms (Gomory, 1958)
- Branch and Bound (Land and Doig, 1960)

However,

solution methods to LP problems with integer or Boolean variables are still far less efficient than those which include continuous variables only

Example Problem

Maximize $\mathbf{Z} = 5x_1 + 2x_2 + x_3$

subject to

$$x_1 + 3x_2 - x_3 \leq 6,$$

$$x_2 + x_3 \leq 4,$$

$$3x_1 + x_2 \leq 7,$$

$$x_1, x_2, x_3 \geq 0.$$

Simplex and Example Problem

Step 1. Convert to Standard Form

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1, \qquad a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n + x_{n+1} = b_1,$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \ge b_2, \quad \Longrightarrow \quad a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n - x_{n+2} = b_2,$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m, \qquad a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n + x_{n+k} = b_m,$$

In our example problem:

$$x_1 + 3x_2 - x_3 \le 6, \qquad x_1 + 3x_2 - x_3 + x_4 = 6,$$

$$x_2 + x_3 \le 4, \quad \Longrightarrow \quad x_2 + x_3 + x_5 = 4,$$

$$3x_1 + x_2 \le 7, \qquad 3x_1 + x_2 + x_6 = 7,$$

$$x_1, x_2, x_3 \ge 0. \qquad x_1, x_2, x_3, x_4, x_5, x_6 \ge 0.$$

Simplex: Step 2

Step 2. Start with an initial basic feasible solution (b.f.s.) and set up the initial tableau.

In our example

Maximize **Z =** $5x_1 + 2x_2 + x_3$

$x_1 + 3x_2 - x_3 + x_4 = 6,$

$x_2 + x_3 + x_5 = 4,$

$3x_1 + x_2 + x_6 = 7,$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	1	3	-1	1	0	0	6
0	x_5	0	1	1	0	1	0	4
0	x_6	3	1	0	0	0	1	7
$\bar{c}$ row		5	2	1	0	0	0	Z=0

Step 2: Explanation

Adjacent Basic Feasible Solution

If we bring a nonbasic variable x_s into the basis, our system changes from the basis, x_b, to the following (same notation as the book):

$$x_1 + \bar{a}_{1s}x_s = \bar{b}_1$$

$$\vdots$$

$$x_r + \bar{a}_{rs}x_r = \bar{b}_r$$

$$\vdots$$

$$x_m + \bar{a}_{ms}x_s = \bar{b}_s$$

$x_i = \bar{b}_i - \bar{a}_{is}$ for i =1, …, m

$x_s = 1$

$x_j = 0$ for j=m+1, ..., n and $j \neq s$

The new value of the objective function becomes:

$$Z = \sum_{i=1}^{m} c_i(\bar{b}_i - \bar{a}_{is}) + c_s$$

Thus the change in the value of Z per unit increase in x_s is

$$\bar{c}_s = \text{new value of Z - old value of Z}$$

$$= \sum_{i=1}^{m} c_i(\bar{b}_i - \bar{a}_{is}) + c_s - \sum_{i=1}^{m} c_i\bar{b}_i$$

$$= c_s - \sum_{i=1}^{m} c_i\bar{a}_{is}$$

This is the Inner Product rule

Simplex: Step 3

Use the inner product rule to find the relative profit coefficients

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	1	3	-1	1	0	0	6
0	x_5	0	1	1	0	1	0	4
0	x_6	3	1	0	0	0	1	7
$\bar{c}$ row		5	2	1	0	0	0	Z=0

$$\bar{c}_j = c_j - c_B \overline{P}_j$$

$\underline{c}_1$ = 5 - 0(1) - 0(0) - 0(3) = 5 -> largest positive

$\underline{c}_2$ = ….

$\underline{c}_3$ = ….

Step 4: Is this an optimal basic feasible solution?

Simplex: Step 5

Apply the minimum ratio rule to determine the basic variable to leave the basis.

The new values of the basis variables:

$$x_i = \bar{b}_i - \bar{a}_{is} x_s \qquad \text{for } i = 1, \ldots, m$$

$$\max x_s = \min_{\bar{a}_{is} > 0} \left[\frac{\bar{b}_i}{\bar{a}_{is}} \right]$$

In our example:

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	1	3	-1	1	0	0	6
0	x_5	0	1	1	0	1	0	4
0	x_6	3	1	0	0	0	1	7
$\bar{c}$ row		5	2	1	0	0	0	Z=0

Row	Basic Variable	Ratio
1	x_4	6
2	x_5	-
3	x_6	7/3

Simplex: Step 6

Perform the pivot operation to get the new tableau and the b.f.s.

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	1	3	-1	1	0	0	6
0	x_5	0	1	1	0	1	0	4
0	x_6	3	1	0	0	0	1	7
$\bar{c}$ row		5	2	1	0	0	0	Z=0

New iteration: find entering variable:

$$\bar{c}_j = c_j - c_B \bar{P}_j$$

$c_B = (0\ 0\ 5)$

$c_2 = 2 - (0)\ 8/3 - (0)\ 1 - (5)\ 1/3 = 1/3$
$c_3 = 1 - (0)\ (-1) - (0)\ 1 - (5)\ 0 = 1$
$c_6 = 0 - (0)\ 0 - (0)\ 0 - (5)\ 1/3 = -5/3$

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	8/3	-1	1	0	0	11/3
0	x_5	0	1	1	0	1	0	4
5	x_1	1	1/3	0	0	0	1/3	7/3
$\bar{c}$ row		0	1/3	1	0	0	-5/3	Z=35/3

Final Tableau

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	8/3	-1	1	0	0	11/3
0	x_5	0	1	1	0	1	0	4
5	x_1	1	1/3	0	0	0	1/3	7/3
$\bar{c}$ row		0	1/3	1	0	0	-5/3	Z=35/3

x_3 enters basis,
x_5 leaves basis

Wrong value!
4 should be 11/3

c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	4	0	1	1	0	23/3
1	x_3	0	1	1	0	1	0	4
5	x_1	1	1/3	0	0	0	1/3	7/3
$\bar{c}$ row		0	-2/3	0	0	-1	-5/3	Z=47/3

Optimal solution obtained: $z = 47/3$,
$x_1=7/3, x_3=4, x_2 = x_4 = 0 = x_3$

Assignment

- Try yourself

Q.1 Solve the LPP by simplex method:

Maximize $z = 5x_1 + 4x_2 + 3x_3$

Subject to the constraints

$$3x_1 + 2x_2 + x_3 \le 10, \quad 2x_1 + x_2 + 2x_3 \le 12, \quad x_1 + x_2 + 3x_3 \le 15$$

$$x_1, x_2, x_3 \ge 0$$

Q.2 Solve the LPP by simplex method:

Maximize $z = 5x_1 + 3x_2$

Subject to $3x_1 + x_2 \le 2, \quad 5x_1 + 2x_2 \ge 10, \quad 3x_1 + 8x_2 \le 12$

$x_1, x_2 \ge 0$

Q.3: Solve the LPP by simplex method:

Maximize $z = 10x_1 + x_2 + 2x_3$

Subject to the constraints

$$3x_1 + x_2 - 2x_3 \le 10, \quad 4x_1 + x_2 + x_3 \le 20,$$

$$x_1, x_2, x_3 \ge 0$$

- **Thank you**